

When the Cut Condition is Enough: A Complete Characterization for Multiflow Problems in Series-Parallel Networks*

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Abstract

Let $G = (V, E)$ be a supply graph and $H = (V, F)$ a demand graph defined on the same set of vertices. An assignment of capacities to the edges of G and demands to the edges of H is said to satisfy the *cut condition* if for any cut in the graph, the total demand crossing the cut is no more than the total capacity crossing it. The pair (G, H) is called *cut-sufficient* if for any assignment of capacities and demands that satisfy the cut condition, there is a multiflow routing the demands defined on H within the network with capacities defined on G .

We prove a previous conjecture, which states that when the supply graph G is series-parallel, the pair (G, H) is cut-sufficient if and only if (G, H) does not contain an *odd spindle* as a minor; that is, if it is impossible to contract edges of G and delete edges of G and H so that G becomes the complete bipartite graph $K_{2,p}$, with $p \geq 3$ odd, and H is composed of a cycle connecting the p vertices of degree 2, and an edge connecting the two vertices of degree p . We further prove that if the instance is *Eulerian* — that is, the demands and capacities are integers and the total of demands and capacities incident to each vertex is even — then the multiflow problem has an integral solution. We provide a polynomial-time algorithm to find an integral solution in this case.

In order to prove these results, we formulate properties of tight cuts (cuts for which the cut condition inequality is tight) in cut-sufficient pairs. We believe these properties might be useful in extending our results to planar graphs.

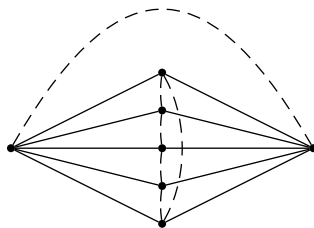


Figure 1: A 5-spindle. Supply edges are solid and demand edges are dashed.

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1 Introduction

When does a network admit a flow that satisfies a given collection of point-to-point demands? This broad question has led to a number of important results over the last several decades. The most fundamental of these considers the case of a *single* demand, from a source vertex to a sink vertex. In this case, the network is able to satisfy the demand if and only if for every cut separating the source from the sink, the total capacity of network edges crossing the cut is no less than the demand: this holds regardless of the topology of the network. This is the famous max-flow min-cut theorem, celebrated both for its elegance and its very wide applicability across computer science, graph theory, and operations research.

Things get much more interesting, and intricate, when we generalize to the multicommodity case. It is easy to see that in order to have a flow satisfying all demands, it is necessary that for all cuts the total capacity crossing the cut is no less than the total demand crossing it. This is called the *cut condition*. Unlike in the single-commodity case, this is no longer a sufficient condition in general [14]. This has led to two kinds of generalizations: (1) finding conditions on the topology of the network and/or the structure of the demands that make the cut condition sufficient, and (2) understanding how “far” from sufficient the cut condition can be. We shall discuss both categories of results below, after the necessary basic definitions. The work presented in this paper falls into the first category.

The simplest example demonstrating that the cut condition does not suffice is the network $K_{2,3}$, with unit capacities and unit demand between each pair of non-adjacent vertices. This example has a natural generalization to the network $K_{2,p}$ for odd $p \geq 3$, as suggested by Figure 1. Our main theorem says that for the important class of series-parallel networks, these examples — which we call *odd spindles* — are (in a sense) the *only* ones where the cut condition does not suffice.

The single-commodity flow problem has another nice property that does not extend to the multicommodity case: if the demand and the capacities are integers and the network can satisfy the demand, then it can do so with an *integral* flow. In our work, we show that integral multicommodity flow instances on series-parallel networks that satisfy the cut condition and avoid the above odd spindles admit *half-integral* flows satisfying the demands (in fact, we show a stronger result which implies this; see below). Moreover, for such instances, we give a polynomial time algorithm to compute such a flow.

1.1 Basic Definitions and Background

Given an undirected graph $G = (V, E)$, with capacities c_e on the edges $e \in E$, let \mathcal{P} be the set of simple paths in G . A *multiflow* is an assignment $f : \mathcal{P} \rightarrow \mathbb{R}_+$. It is said to be *feasible* if, for each $e \in E$, we have $\sum_{P \in \mathcal{P}(e)} f_P \leq c_e$, where $\mathcal{P}(e)$ is the set of paths in G that contain the edge e . Let $H = (V, F)$ be another graph on the same set of vertices, with demands D_i on the edges $i \in F$. The multiflow f is said to *satisfy* H if for each edge $i \in F$, we have $\sum_{P \in \mathcal{P}[i]} f_P \geq D_i$, where $\mathcal{P}[i]$ is the set of paths in G that connect the endpoints of i . The tuple (G, H, c, D) forms an instance of the multiflow (or multicommodity flow) problem, which consists of finding whether there exists a feasible multiflow in G satisfying H ; if so, the instance is said to be *routable*. We call G and H the *supply graph* and *demand graph* (respectively) of the instance.

For each set $C \subseteq V$, the *cut* $\delta_G(C)$ generated by C in G is defined to be the set of edges in G with exactly one endpoint in C . We define $\delta_H(C)$ similarly. The *surplus* $\sigma(C)$ of C is the total capacity of the edges in $\delta_G(C)$ minus the total demand of the edges in $\delta_H(C)$: $\sigma(C) = \sum_{e \in \delta_G(C)} c_e - \sum_{i \in \delta_H(C)} D_i$. The cut condition is then the statement that every cut has nonnegative surplus: $\sigma(C) \geq 0$ for all $C \subseteq V$. As noted above, an instance (G, H, c, D) must satisfy the cut condition in order to be routable. Our goal is to understand when this condition is sufficient.

The graph pair (G, H) is *cut-sufficient* if for all assignments of capacities c and demands D that satisfy the cut condition, the instance (G, H, c, D) is routable. One of the earliest cut-sufficiency theorems is due to Hu [7] and states that (G, H) is cut-sufficient if H is the union of two stars, i.e., if all of its edges can be

covered by two vertices. Notice that this theorem applies to a general G , but it greatly restricts H . Network flow literature abounds with other cut-sufficiency theorems [7, 11, 13, 14, 16, 18, 19]. Many of these impose conditions on both G and H ; a well-known example is the Okamura-Seymour Theorem, which states that a pair (G, H) is cut-sufficient if G is planar and all edges of H have their endpoints on a single face of G [14]. Schrijver [17, Chapter 70] surveys several cut-sufficiency theorems and many related concepts and topics.

1.2 Our Contributions

We give a sharp characterization of cut-sufficient graph pairs where the supply graph is *series-parallel*. Further, for integral multiflow instances on such cut-sufficient pairs, we show that the cut condition together with a natural “Eulerian” condition imply that a feasible integral solution exists; we also give a polynomial time algorithm to find an integral solution. Finally, our work here suggests to us a conjecture that would characterize cut-sufficiency in *planar* graphs. The details follow.

We define a p -*spindle* to be a pair of graphs (G, H) such that the supply graph G is $K_{2,p}$, with $p \geq 3$, and the demand graph H consists of a cycle connecting the p vertices of degree 2 in G , and an additional demand edge between the two remaining vertices. An *odd spindle* is a p -spindle with p odd.

Theorem 1.1 (Fractional Routing Theorem; characterization of cut-sufficiency) *If the supply graph G is series-parallel, then the pair (G, H) is cut-sufficient if and only if the pair (G, H) cannot be reduced to an odd spindle by contraction of edges of G and deletion of edges of G and H .*

Schrijver [17, Section 70.11] gives a number of sufficient conditions for cut-sufficiency; our characterization above is sharper than all of these when G is series-parallel. The above result was conjectured in Chekuri et al. [3, Conjecture 3.5]. To prove it, we first revisit the connection between multiflow problems and metric embeddings via linear programming duality. Unlike previous works that used this approach, we exploit *complementary slackness* to derive some LP-based conditions for cut-sufficiency, in Section 3. These conditions do not refer to the structure of G , and so could be useful in extending our results from series-parallel graphs to more general classes. The proof of Theorem 1.1 itself appears in Section 4.

We say that an instance (G, H, c, D) is *Eulerian* if all capacities c_e and demands D_i are integers and $\sigma(C)$ is even for all $C \subseteq V$; recall that σ depends on c and D .

Theorem 1.2 (Integral Routing Theorem) *If G is series-parallel, (G, H) is cut-sufficient, and the multiflow instance (G, H, c, D) satisfies the cut condition and is Eulerian, then the problem has an integral solution. Moreover, under these conditions, an integral solution can be computed in polynomial time.*

This implies that under the same assumptions except for the Eulerian condition, the multiflow problem has a half-integral solution. Similar uses of the Eulerian condition are ubiquitous in the literature [13, 14, 16]. We prove the above result in Section 5. The algorithm is described in Section 5.1.

Planar supply graphs allow one other obstruction to cut-sufficiency, apart from the odd spindles. We conjecture, in Section 6, that there are no further examples: this would extend our results to instances where G is planar.

1.3 Other Related Work

A different approach to the relation between multiflows and cuts was pioneered by Leighton and Rao [9], who sought to understand how “far” from sufficient the cut condition could be. To be precise, let us define the *maximum concurrent flow* for a multiflow instance (G, H, c, D) to be the largest fraction ϕ such that $(G, H, c, \phi D)$ is routable. In this paper, we adopt the equivalent approach of studying the *minimum congestion* $\alpha \geq 1$ such that $(G, H, \alpha c, D)$ is routable: it is easy to see that $\phi = 1/\alpha$ for any instance. For a pair of

graphs (G, H) , the *flow-cut gap* is defined as the maximum, over all choices of demands and capacities that satisfy the cut condition, of the minimum congestion. The larger this gap the further the pair (G, H) is from cut-sufficiency. Clearly, a pair is cut-sufficient if and only if its flow-cut gap is 1.

There has been intense research on finding the flow-cut gaps for various classes of graphs, a line of work originally motivated by the problem of approximating sparsest cuts [1, 2, 5, 6, 10]. The class of series-parallel instances is notable, as it is one of the very few classes for which there are precise bounds on the flow-cut gap: Chakrabarti et al. [2] show that the gap cannot be more than 2, whereas Lee and Raghavendra [8] show that it can be as close to 2 as desired. Chekuri et al. [4] show that series-parallel instances have *integral multiflows* that do not use more than 5 times the capacity of the supply graph. A special case of the integer multiflow problem is the *disjoint paths problem*, where $D_i = 1$ for all i and $c_e = 1$ for all e . In general, the disjoint paths problem is NP-complete even when restricted to series-parallel graphs [12].

The seminal work of Linial et al. [10] connected flow-cut gaps to metric embeddings via LP duality: we now briefly explain this connection, which we also use in our work. Every positive length function l on the edges of a graph determines a *shortest-path metric*, which is a distance function d on the vertices of the graph, such that $d(u, v) = \min_{P \in \mathcal{P}[u, v]} \sum_{e \in P} l_e$; here $\mathcal{P}[u, v]$ denotes the set of paths between the vertices u and v . Every assignment of non-negative real values x_C to subsets C of vertices of a graph determines a *cut-cone metric*, which is a distance function d on the vertices of the graph defined by $d(u, v) = \sum_{C: \{u, v\} \cap C = 1} x_C$. For any two distance functions d and d' defined on the vertices of a graph such that $d \geq d'$, the *distortion* from d to d' is defined to be $\max_{u \neq v} d(u, v)/d'(u, v)$. For a distance function d and a family of metrics \mathcal{M} , the minimum distortion embedding of d into \mathcal{M} is a distance function d' in \mathcal{M} that minimizes the distortion from d to d' . Linial et al. [10] show that the maximum congestion required for a particular supply graph G equals the maximum distortion required to embed any possible shortest-path metric on G into the family of cut-cone metrics.¹ We shall call this the *congestion-distortion equivalence* theorem.

2 Definitions and Preliminaries

A subset of vertices $C \subseteq V$ and the corresponding cut $\delta_G(C)$ are called *central* if both C and $V \setminus C$ are connected in G . It is well-known and easy to prove that if the surplus σ is nonnegative for all central cuts, then the cut condition is satisfied [17, Theorem 70.4]. A subset C and the cut $\delta_G(C)$ are *tight* if $\sigma(C) = 0$.

We assume in this article that the supply graph G is biconnected. It is not hard to show that if G is not biconnected, the multiflow problem can be solved separately on its biconnected components. A biconnected graph is *series-parallel* if and only if it does not contain K_4 as a minor. A pair of graphs (G, H) is series-parallel if the supply graph G is series-parallel.

We use an extension of graph minors to pairs (G, H) of supply and demand graph, as proposed in [4].

Definition 2.1 *Let (G, H) and (G', H') be two pairs of graphs. Then (G', H') is a minor of (G, H) if we can obtain (G', H') from (G, H) by contracting and deleting edges of G , and deleting edges of H .*

Here, *deleting* an edge means removing it from the graph, and *contracting* an edge means removing it and merging its endpoints.

2.1 Surplus Identities

Recall that the *surplus* $\sigma(X)$ of $X \subseteq V$ is the total capacity minus the total demand crossing the cut $\delta_G(X)$. Additionally, for X and Y disjoint, let $\delta_G(X, Y)$ and $\delta_H(X, Y)$ be the set of edges in G , respectively H , with one

¹It is a simple exercise to show that the family of cut-cone metrics coincides with that of ℓ_1 -embeddable metrics.

endpoint in X and one in Y , and let $\sigma(X, Y) := \sum_{e \in \delta_G(X, Y)} c_e - \sum_{e \in \delta_H(X, Y)} D_i$. In particular, $\sigma(X, V \setminus X) = \sigma(X)$. The surplus function σ satisfies the following useful identities.

Lemma 2.2 *Let A and B be two subsets of V . Then*

(a) If B_1, \dots, B_k is a partition of B , then $\sigma(A, B) = \sigma(A, B_1) + \dots + \sigma(A, B_k)$.

(b) In particular, if $B = V \setminus A$, then $\sigma(A) = \sigma(A, B_1) + \dots + \sigma(A, B_k)$.

(c) $\sigma(A \cup B) + \sigma(A \cap B) = \sigma(A) + \sigma(B) - 2\sigma(A \setminus B, B \setminus A)$,

(d) $\sigma(A \setminus B) + \sigma(B \setminus A) = \sigma(A) + \sigma(B) - 2\sigma(A \cap B, V \setminus (A \cup B))$.

(e) In particular, if A and B are disjoint, then $\sigma(A \cup B) = \sigma(A) + \sigma(B) - 2\sigma(A, B)$.

Proof: (a) and (b) are easy to prove, and are left as an exercise. Let us use \overline{X} to denote $V \setminus X$, for each subset $X \subseteq V$. By (a) and (b), we have

$$\begin{aligned}\sigma(A \cup B) &= \sigma(A \cup B, \overline{A \cup B}) = \sigma(A \setminus B, \overline{A \cup B}) + \sigma(A \cap B, \overline{A \cup B}) + \sigma(B \setminus A, \overline{A \cup B}), \\ \sigma(A \cap B) &= \sigma(A \cap B, \overline{A \cap B}) = \sigma(A \cap B, \overline{A \cup B}) + \sigma(A \cap B, A \setminus B) + \sigma(A \cap B, B \setminus A), \\ \sigma(A) &= \sigma(A, \overline{A}) = \sigma(A \cap B, \overline{A \cup B}) + \sigma(A \cap B, B \setminus A) + \sigma(A \setminus B, \overline{A \cup B}) + \sigma(A \setminus B, B \setminus A), \\ \sigma(B) &= \sigma(B, \overline{B}) = \sigma(A \cap B, \overline{A \cup B}) + \sigma(A \cap B, A \setminus B) + \sigma(B \setminus A, \overline{A \cup B}) + \sigma(B \setminus A, A \setminus B).\end{aligned}$$

Simplifying, we get (c). Additionally, we have

$$\begin{aligned}\sigma(A \setminus B) &= \sigma(A \setminus B, \overline{A \setminus B}) = \sigma(A \setminus B, \overline{A \cup B}) + \sigma(A \setminus B, B \setminus A) + \sigma(A \setminus B, A \cap B), \\ \sigma(B \setminus A) &= \sigma(B \setminus A, \overline{B \setminus A}) = \sigma(B \setminus A, \overline{A \cup B}) + \sigma(B \setminus A, A \setminus B) + \sigma(B \setminus A, A \cap B).\end{aligned}$$

By comparing to the equations for $\sigma(A)$ and $\sigma(B)$, we get (d). Finally, (e) is just a restatement of (c) for A and B disjoint. ■

2.2 Properties of Biconnected and Series-Parallel Graphs

We now establish a number of simple but useful properties of biconnected and series-parallel graphs that arise at various points in the proofs of our main theorems. The reader who wishes to focus on the main theorems may safely skip to Section 3.

Lemma 2.3 *In a series-parallel graph, a simple cycle does not intersect any central cut more than twice.*

Proof: A cycle intersects any cut an even number of times. Suppose a cycle Q intersects a cut $\delta_G(C)$ four times or more. Then we can choose four vertices u_1, u_2, u_3, u_4 in order on Q such that u_1 and u_3 are on one side of $\delta_G(C)$ and u_2 and u_4 on the other. Then there is a path connecting u_1 to u_3 on one side of $\delta_G(C)$, and a path connecting u_2 to u_4 on the other. This creates a K_4 minor, which cannot exist in a series-parallel graph. ■

Lemma 2.4 *In a biconnected graph G , for any three distinct vertices s, u and t , there is a simple path from s to t containing u .*

Proof: Since G is biconnected, there are two vertex-disjoint paths P_1 and P_2 from s to u . Biconnectivity also implies there is a path P from u to t disjoint from s . If P does not intersect P_1 (or P_2), then P_1 (or P_2) followed by P creates a simple path connecting s – u – t in that order. Otherwise, let w be the last vertex of P , from u to t , that is in P_1 or P_2 . Since P_1 and P_2 are disjoint, w is in only one of them, say P_2 . Then P_1 , the part of P_2 from u to w , and the part of P from w to t is simple, and connects s – u – t in that order. ■

Lemma 2.5 *In a biconnected graph G , for any two vertices s and t and any edge (u, v) , there is a simple path from s to t containing the edge (u, v) .*

Proof: By Lemma 2.4, there are simple paths P_{sut} and P_{svt} from s to t containing u and v respectively. If P_{sut} contains v , or P_{svt} contains u , then using the edge (u, v) to shortcut the path, we get a path from s to t containing (u, v) . Let P_{su}, P_{sv}, P_{ut} and P_{vt} be the subpaths of P_{sut} and P_{svt} between corresponding vertices. In the set of vertices in $P_{sut} \cap P_{svt}$, let w be a vertex closest to u on P_{sut} (in P_{su} or P_{ut}). Without loss of generality, suppose that w is in P_{sv} . Then let P_{sw} be the subpath of P_{sv} from s to w , and P_{wu} the subpath of P_{sut} from w to u . The path $P_{sw} \cup P_{wu} \cup (u, v) \cup P_{vt}$ goes from s to t and contains (u, v) . ■

In a biconnected series-parallel graph G , a pair of vertices (s, t) is a *split pair* if the graph G remains series-parallel after adding an edge from s to t . In an oriented graph, a *source* is a vertex that has only outgoing edges, and a *sink* is a vertex that has only incoming edges.

Lemma 2.6 *In a biconnected series-parallel graph G , for any split pair (s, t) , there is a unique way of orienting the edges of G such that G is acyclic, and s and t are the unique source and sink respectively. This orientation has the property that any simple path from s to t is oriented, and any oriented path can be extended into an oriented path from s to t .*

Proof: Let (u, v) be any edge in G . By Lemma 2.5, there is at least one simple path from s to t containing (u, v) . Suppose there are two such paths P_1 and P_2 , connecting $s-u-v-t$ and $s-v-u-t$ in these orders respectively. These two paths plus an (s, t) edge create a K_4 minor, which contradicts the fact that (s, t) is a split pair. Therefore, there are either only paths connecting $s-u-v-t$ in that order, or only paths connecting $s-v-u-t$ in that order. We orient the edge (u, v) in the order given by these paths. Trivially, any path from s to t is oriented.

We claim that orienting all edges in this way creates an acyclic orientation such that s and t are the unique source and sink respectively. Suppose that some vertex $u \neq s$ is a source. For any edge (u, v) , there is a simple path connecting $s-u-v-t$ in that order. Therefore this path is oriented, and so u is not a source, a contradiction. Thus, s is indeed the unique source. Symmetrically, t is the unique sink. Suppose that the orientation creates an oriented cycle. For any edge (u, v) in the cycle, there is a simple path P connecting $s-u-v-t$ in that order. Let w and z be the first and last vertex of the cycle in P . The cycle creates two paths from w to z , one whose orientation must be inconsistent with the path connecting $s-w-z-t$ in that order; and so there are no oriented cycles.

Finally, for any oriented path, it is possible to extend it into an oriented path from s to t by adding edges at the beginning until it starts from s , and at the end until it ends at t . ■

For an orientation of G defined by a split pair (s, t) , if there is an oriented path from u to v , then (u, v) is *compliant*. For any non-compliant pair of vertices (u, v) , let P_{sut} and P_{svt} be two oriented paths from s to t containing u and v respectively. Let P_{su}, P_{sv}, P_{ut} and P_{vt} be the subpaths of P_{sut} and P_{svt} connecting the two corresponding vertices. The pair (w, z) is called the *terminals of (u, v)* if w is the last common vertex of P_{su} and P_{sv} , and z is the first common vertex of P_{ut} and P_{vt} . We prove now that the pair (w, z) is independent of the choice of P_{sut} and P_{svt} . We say a pair of vertices (w, z) *separates* vertices u from v if u and v are in different connected components of $V \setminus \{w, z\}$.

Lemma 2.7 *For any non-compliant pair (u, v) , there is a unique pair (w, z) of terminals of (u, v) . The pair (w, z) is a 2-vertex-cut separating u from v . Furthermore, unless s is w , (w, z) separates u and v from s , and unless t is z , (w, z) separates u and v from t . Any simple cycle containing u and v also contains w and z , is composed of two oriented paths from w to z , and has w as unique source and z as unique sink.*

Proof: By Lemma 2.4, there are simple paths P_{sut} and P_{svt} from s to t containing u and v respectively; by Lemma 2.6, these paths are oriented. So (u, v) always has at least one pair (w, z) of terminals. Since (s, t) is a split pair, we can assume there is an (s, t) edge and still have G series-parallel. Then there are at least

three vertex-disjoint paths from w to z , one through u , one through v , and one containing (s, t) . So any path connecting vertices from two of these three paths must contain w or z , because otherwise the graph would contain a K_4 minor. This means that (w, z) is a 2-vertex-cut separating u from v ; and if s is not w or t is not z , then they are also separated from u and v by (w, z) . This is true for any pair of terminals of (u, v) .

Let C be any simple cycle containing u and v . Since (w, z) separates u from v , C must contain w and z . Since a simple cycle can intersect only two connected components of $G \setminus \{w, z\}$, C does not contain s or t , unless they are w or z respectively. So C is composed of two oriented paths from w to z , containing u and v respectively. And so w and z are the unique source and sink of C .

Since any simple cycle containing u and v also contains all the pairs of terminals of (u, v) , and has any pair of terminals as unique source and unique sink, there is only one pair of terminals of (u, v) . ■

For any orientation of G defined by a split pair (s, t) , a pair of vertices (w, z) is said to *bracket* another pair (u, v) if there is an oriented path from w to z containing u and v . If (w, z) brackets (u, v) but $w \neq u$ or $z \neq v$, then (w, z) *strictly brackets* (u, v) . Since the orientation is acyclic, the bracketing relation is transitive.

Lemma 2.8 *Let $G = (V, E)$ be a series-parallel graph, and let $u, v \in V$ be two arbitrary vertices. Then G can be embedded in the plane so that both u and v are on the outside face.*

Proof: If G is series-parallel, then it does not contain a K_4 minor. Hence adding any single edge e to G does not create either a K_5 or a $K_{3,3}$ minor; it follows that adding e to G results in a planar graph. In particular, $G' = (V, E \cup \{(u, v)\})$ is planar. We embed G' in the plane so that (u, v) is on the outside face. (See, e.g., [15].) Removing (u, v) from the result gives an embedding of G with u and v on the outside face. ■

3 Congestion-Distortion Equivalence via LP Duality and Consequences

We now give our new proof of the congestion-distortion equivalence theorem (see Section 1), using only basic notions of linear programming duality. Our proof will reveal several additional relations between LP variables that are useful later: in particular, they give us cut-sufficiency conditions based on certain LP variables. The starting point of the proof is a well-known fact: multiflows are tightly related to metrics, because the dual of the LP expressing a multiflow problem can be interpreted as the problem of finding a certain graph metric.

3.1 The Proof via LP Duality

Throughout this section, we fix a “supply graph” $G = (V, E)$ and a “demand graph” $H = (V, F)$. The crux of the proof is to identify a certain nonlinear maximization problem (+) in variables $c = \{c_e\}_{e \in E}$, $D = \{D_i\}_{i \in F}$, $l = \{l_e\}_{e \in E}$, and $d = \{d_i\}_{i \in F}$ that has the following two properties. First, for each setting of c and D satisfying the cut condition, the program (+) reduces to a maximization LP whose dual is the problem of finding the minimum congestion for the multiflow problem (G, H, c, D) . Second, for each setting of l and d satisfying certain metric inequalities, the program (+) reduces to a different maximization LP whose dual is (a generalization of) the problem of finding the minimum distortion embedding, into the family of cut-cone metrics, of the metric given by l and d . It follows that the maximum possible congestion over all capacity/demand settings equals the maximum possible distortion over all length settings. We now give the details.

For each $i \in F$, let $\{P_1^i, P_2^i, \dots\}$ be a listing of $\mathcal{P}[i]$, the set of simple paths in G connecting the endpoints of the demand i . The problem of determining the minimum congestion for the multiflow instance (G, H, c, D)

can be written as²

$$\begin{aligned}
z(c, D) = \min \quad & \alpha \\
s.t. \quad & \sum_{i,j: e \in P_j^i} \mathbf{f}_j^i \leq c_e \alpha \quad \forall e \in E \\
& \sum_j \mathbf{f}_j^i \geq D_i \quad \forall i \in F \\
& \mathbf{f}_j^i \geq 0 \quad \forall i, j,
\end{aligned} \tag{P}$$

where \mathbf{f}_j^i is the variable indicating the amount routed on path P_j^i . The dual linear program is the following:

$$\begin{aligned}
z(c, D) = \max \quad & \sum_i D_i \mathbf{d}_i \\
s.t. \quad & \sum_e c_e \mathbf{l}_e = 1 \\
& \mathbf{d}_i \leq \sum_{e \in P_j^i} \mathbf{l}_e \quad \forall i, j \\
& \mathbf{d}_i \geq 0 \quad \forall i \in F \\
& \mathbf{l}_e \geq 0 \quad \forall e \in E.
\end{aligned} \tag{D}$$

The variables \mathbf{l}_e can be thought of as *lengths* of the edges of G , and \mathbf{d}_i as *distances* between the endpoints of i . The second set of constraints are *metric inequalities*, which ensure that \mathbf{d}_i is no more than the shortest-path distance between the endpoints of i induced by the lengths \mathbf{l}_e .

In order to find the flow-cut gap of a pair (G, H) , we need to find the maximum value to (P) (and (D)) over all choices of capacities c and demands D that satisfy the cut condition, which can be expressed as

$$\begin{aligned}
\max \quad & z(\mathbf{c}, \mathbf{D}) \\
s.t. \quad & \sum_{i \in \delta_H(C)} \mathbf{D}_i \leq \sum_{e \in \delta_G(C)} \mathbf{c}_e \quad \forall C \subseteq V \\
& \mathbf{D}_i \geq 0 \quad \forall i \in F, \\
& \mathbf{c}_e \geq 0 \quad \forall e \in E.
\end{aligned} \tag{1}$$

Since the linear program (D) is a maximization problem, we can write the problem of finding the flow-cut gap as a single maximization problem on variables $\mathbf{c}, \mathbf{D}, \mathbf{l}$ and \mathbf{d} :

$$\begin{aligned}
\max \quad & \sum_i \mathbf{D}_i \mathbf{d}_i \\
s.t. \quad & \sum_e \mathbf{c}_e \mathbf{l}_e = 1 \\
& \mathbf{d}_i \leq \sum_{e \in P_j^i} \mathbf{l}_e \quad \forall i, j \\
& \sum_{i \in \delta_H(C)} \mathbf{D}_i \leq \sum_{e \in \delta_G(C)} \mathbf{c}_e \quad \forall C \subseteq V \\
& \mathbf{D}_i, \mathbf{d}_i \geq 0 \quad \forall i \in F \\
& \mathbf{c}_e, \mathbf{l}_e \geq 0 \quad \forall e \in E.
\end{aligned} \tag{+}$$

This is not a linear program, since some of the variables multiply each other. However, there are two ways we can transform it into a linear program by setting some variables to be parameters. If we fix \mathbf{c}_e for all e and \mathbf{D}_i for all i to be parameters that satisfy the cut condition, we obtain the linear program (D). But if we

²In this section, boldface is used to distinguish variables from parameters in the linear programs.

fix \mathbf{l}_e for all e and \mathbf{d}_i for all i to be parameters that satisfy the metric inequalities, we find a different linear program in variables \mathbf{c}_e and \mathbf{D}_i :

$$\begin{aligned}
w(l, d) = \max \quad & \sum_i d_i \mathbf{D}_i \\
s.t. \quad & \sum_e l_e \mathbf{c}_e = 1 \\
& \sum_{i \in \delta_H(C)} \mathbf{D}_i \leq \sum_{e \in \delta_G(C)} \mathbf{c}_e \quad \forall C \subseteq V \\
& \mathbf{D}_i \geq 0 \quad \forall i \in F \\
& \mathbf{c}_e \geq 0 \quad \forall e \in E.
\end{aligned} \tag{D'}$$

The flow-cut gap problem (1) and (+) can then also be expressed as

$$\begin{aligned}
\max \quad & w(\mathbf{l}, \mathbf{d}) \\
s.t. \quad & \mathbf{d}_i \leq \sum_{e \in P_j^i} \mathbf{l}_e \quad \forall i, j \\
& \mathbf{d}_i \geq 0 \quad \forall i \in F \\
& \mathbf{l}_e \geq 0 \quad \forall e \in E.
\end{aligned} \tag{2}$$

Notice that in a solution achieving the maximum above, each \mathbf{d}_i must equal the shortest-path distance between the endpoints of i induced by the lengths \mathbf{l}_e . The dual of (D') is

$$\begin{aligned}
w(l, d) = \min \quad & \gamma \\
s.t. \quad & \sum_{C: e \in \delta_G(C)} \mathbf{x}_C \leq l_e \gamma \quad \forall e \in E, \\
& \sum_{C: i \in \delta_H(C)} \mathbf{x}_C \geq d_i \quad \forall i \in F \\
& \mathbf{x}_C \geq 0 \quad \forall C \subseteq V.
\end{aligned} \tag{P'}$$

The system (P') has a variable \mathbf{x}_C for each subset $C \subseteq V$. The values of these variables define a cut-cone metric; call it d' . The first constraint says that the d' -length of an edge e is at most γ times its “true” length l_e . The second constraint says that the d' -distance between the endpoints of a demand i is at least d_i , which, for l and d achieving the maximum in (2), equals the “true” distance given by l . Thus, (P') can be seen as approximating (at least between endpoints of demands) the shortest-path metric induced by l by a cut-cone metric, within an approximation factor γ as small as possible.

As a clean special case, when H is a complete graph on V , then $d_e = l_e$ for each edge e , and thus the two constraints in (P') say (respectively) that $d \geq d'/\gamma$ and that the distortion from d to d'/γ is at most γ . Thus, d embeds into the family of cut-cone metrics with distortion at most γ . The equivalence of (1), (+), and (2) means that the flow-cut gap of (G, H) is equal to the minimum distortion required to embed an arbitrary shortest-path metric defined on G into the family of cut-cone metrics. This completes the proof of the congestion-distortion equivalence, entirely through basic notions of linear programming.

3.2 Implications of the New Proof

Suppose that, for some pair of graphs (G, H) , with $G = (V, E)$, $H = (V, F)$, we have an optimal solution (c^*, D^*, l^*, d^*) to the nonlinear program (+). By the properties of linear programming duality, there are solutions f^* and x^* to the flow problem (P) and the cut metric problem (P') that satisfy complementary slackness. We call $(c^*, D^*, l^*, d^*, f^*, x^*)$ a *general solution* to the pair (G, H) .

Lemma 3.1 *A general solution satisfies the following properties.*

- (a) If $x_C^* > 0$, then $\sum_{i \in \delta_H(C)} D_i^* = \sum_{e \in \delta_G(C)} c_e^*$. Thus, only tight cuts have positive x -value in (P') .
(b) If $f_j^{i*} > 0$ then $d_i^* = \sum_{e \in P_j^i} l_e^*$; and for each i with $D_i^* > 0$, there is a path $P_{j^*}^i$ for which this is true.*

Proof: *(a)* follows from complementary slackness applied to (P') and (D') . The first part of *(b)* follows from complementary slackness applied to (P) and (D) . The second part of *(b)* follows from the second constraint of (P) . ■

Lemma 3.2 *There is a solution x^* to the problem (P') such that only central cuts have a positive x -value.*

Proof: Suppose that in the optimal solution, $x_C^* > 0$ for some C that can be decomposed into two sets C_1 and C_2 that are not connected by any supply edge. Adding the value of x_C^* to the values of $x_{C_1}^*$ and $x_{C_2}^*$ and setting x_C^* to zero increases the distance, in the cut metric defined by x^* , between all pairs of vertices $u \in C_1$, $v \in C_2$. Since there is no supply edge from u to v , there is no upper constraint on this distance in the linear program (P') ; and so the new solution is still optimal. By induction, there is an optimal solution with $x_C^* = 0$ for any non-central C . ■

We assume from now on that in a general solution, if $x_C^* > 0$ then C is central.³

We define a *simple pair* to be a pair (G, H) that has a general solution such that $c_e^* > 0$ and $l_e^* > 0$ for each e , and $D_i^* > 0$ for each i .

Lemma 3.3 *For each pair (G, H) , there is a simple pair with the same flow-cut gap.*

Proof: Suppose that some edge e has a zero capacity in the optimal solution to $(+)$ (i.e. $c_e^* = 0$). This means that there is no upper constraint on the value of l_e , and so the constraints for paths P_j^i which contain edge e put no restriction on the value of d_i ; the constraints on cuts containing e do not change if c_e is in the expression and equal to zero, or removed from the expression. Thus, deleting the edge e from G does not change constraints on d_i and D_i , and so the flow-cut gap remains the same. Similarly, if $l_e^* = 0$, there is no upper constraint on the value of c_e , and so the constraints for cuts C which contain edge e put no restriction on the value of demands crossing C ; the constraints on paths containing e do not change if l_e is in the expression and equal to zero, or removed from the expression. Thus, contracting the edge e does not change constraints on d_i and D_i , and so the flow-cut gap remains the same. If $D_i^* = 0$ for some i , it makes no difference what constraints are on d_i , and so the flow-cut gap remains the same if the demand i is deleted from H . ■

We assume from now on that (G, H) is a simple pair. In what follows, recall that $\mathcal{P}[i]$ denotes the set of paths in G that connect the endpoints of the demand edge $i \in F$.

Lemma 3.4 *Let (G, H) be a simple pair with general solution $(c^*, D^*, l^*, d^*, f^*, x^*)$. Then*

$$\forall i \in F \forall P \in \mathcal{P}[i] : \sum_{C: i \in \delta_H(C)} x_C^* = d_i^* \leq \sum_{e \in P} l_e^* = \frac{1}{\gamma^*} \sum_{e \in P} \sum_{C: e \in \delta_G(C)} x_C^*, \quad (3)$$

with equality if P is a shortest path for the shortest-path metric defined by l^ , e.g., if the solution has a nonzero flow routing the demand i along P .*

³Another way to do this is to decide from the beginning that the optimization program (1) only has cut condition constraints on central cuts, as this is sufficient for ensuring the cut condition is satisfied, which implies that the linear program (P') only has variables x_C for central cuts.

Proof: If $(c^*, D^*, l^*, d^*, f^*, x^*)$ is a general solution for the simple pair (G, H) , then by definition we have $c_e^* > 0$ for all e and $D_i^* > 0$ for all i . By complementary slackness applied to (P') and (D') , this implies that the inequalities of (P') are all tight:

$$l_e^* \gamma = \sum_{C: e \in \delta_G(C)} x_C^* \quad \forall e \in E,$$

$$d_i^* = \sum_{C: i \in \delta_H(C)} x_C^* \quad \forall i \in F.$$

Notably, this implies that the length of any edge e in the cut-cone metric defined by x^* is always γ^* times l_e^* . These equalities imply the equalities in (3). The inequality holds for all solutions. Lemma 3.1(b) shows that the inequality is tight for at least one path, and hence it is tight for the shortest path. ■

And so, in the metric defined by x^* , for any path P_j^i such that $f_j^{i*} > 0$, the ratio between the sum of lengths of edges in the path and the distance between endpoints of the path is equal to the flow-cut gap.

Theorem 3.5 *Let (G, H) be a simple pair with general solution $(c^*, D^*, l^*, d^*, f^*, x^*)$. Suppose there exist $i \in F$ and $P \in \mathcal{P}[i]$ such that P crosses each tight cut at most once, with tightness defined according to c^* and D^* . Then (G, H) is cut-sufficient, i.e., its flow-cut gap is one. More explicitly, a multiflow problem on (G, H) has a fractional solution for any choice of capacities and demands that satisfy the cut condition.*

Proof: Pick a tight set $C \subseteq V$. The number of times that P crosses C is odd if $i \in \delta_H(C)$ and even otherwise. Therefore, by the given condition, if $i \notin \delta_H(C)$, then P must not cross C ; otherwise P must cross C exactly once. Recall that, by Lemma 3.1(a), only tight cuts may have non-zero x^* -values. This implies that

$$\sum_{C: i \in \delta_H(C)} x_C^* = \sum_{e \in P} \sum_{C: e \in \delta_G(C)} x_C^*.$$

In view of Lemma 3.4, this means that

$$\gamma \sum_{C: i \in \delta_H(C)} x_C^* \leq \sum_{e \in P} \sum_{C: e \in \delta_G(C)} x_C^* = \sum_{C: i \in \delta_H(C)} x_C^*.$$

Since the pair (G, H) is simple, each l_e^* is non-zero and thus, so is $\sum_{C: i \in \delta_H(C)} x_C^*$. It follows that $\gamma = 1$ and the flow-cut gap is one, as claimed. ■

4 Proof of the Fractional Routing Theorem

In this section, we prove Theorem 1.1. Namely, for series-parallel graphs G , we show that the pair (G, H) is cut-sufficient if and only if it does not contain an odd spindle as a minor. The following special case of this theorem was proven earlier in Chekuri et al. [3], and we use it in our proof.

Theorem 4.1 ([3, Section 3.3]) *Suppose G is $K_{2,m}$, with possibly an additional supply edge between the two vertices not of degree 2. Then (G, H) is cut-sufficient iff it does not contain an odd spindle as a minor.*

The “only if” direction of Theorem 1.1 is easy, and is proven in Section 3.3 of [3]; we reproduce the argument here. An odd spindle itself has a flow-cut gap of more than 1, as can be seen by setting the capacity of all supply edges and the demand of all demand edges to 1. Let a pair (G, H) contain a pair (G', H') as a minor, and let (G', H', c', D') be an instance of the multiflow problem. We assign capacities c and demands D to the pair (G, H) in the following way. To any supply edge or demand edge that is deleted during the

reduction from (G, H) to (G', H') , we assign a capacity or demand of 0. To any supply edge that is contracted, we assign a very large capacity. And to any edge of (G, H) that is still in (G', H') after the reduction, we assign the capacity or demand of the corresponding edge in (G', H') . Since (G', H', c', D') satisfies the cut condition, so does (G, H, c, D) . For any multiflow solving the instance (G, H, c, D) with congestion γ , we build a multiflow solving the instance (G', H', c', D') with the same congestion γ , by sending on each edge of G' the same flow as on the corresponding edge in G . Therefore, the minimum congestion for (G, H) cannot be less than the minimum congestion for (G', H') . And so, a pair (G, H) cannot be cut-sufficient if it has as a minor a pair (G', H') that is not cut-sufficient.

We now prove the “if” direction. Suppose the pair (G, H) has flow-cut gap more than 1. By Lemma 3.3, we may assume that (G, H) is simple. For a demand (u, v) , a *bubble* for (u, v) is a central set defining a tight cut, but containing neither u nor v . The set $\mathcal{P}[u, v]$ (of paths in G between u and v) is *covered by bubbles* if every path in it crosses a bubble at least once. From Theorem 3.5, and a parity argument, we get the following:

Observation 4.2 *If $P \in \mathcal{P}[u, v]$ does not cross any bubble, then P crosses some tight cut $t > 1$ times, where t is odd.* ■

To prove Theorem 1.1, we first prove that if there is a demand (u, v) such that $\mathcal{P}[u, v]$ is covered by bubbles, then the instance must contain an odd spindle as a minor (Lemma 4.3). We then prove that there must be such a demand (Lemma 4.10).

4.1 Coverage by bubbles creates an odd spindle minor

Lemma 4.3 *If there is a demand (u, v) such that $\mathcal{P}[u, v]$ is covered by bubbles, then the instance must contain an odd spindle as a minor.*

Proof: Let $F_{u,v}$ be a minimal family of bubbles covering all simple paths from u to v . We first claim that $|F_{u,v}| \geq 2$. Indeed, if $F_{u,v} = \{B\}$ for a bubble B , then the vertices u and v are in different connected components of $V \setminus B$. This contradicts the fact that B is central. We now distinguish the following two cases: $|F_{u,v}| \geq 3$, or $|F_{u,v}| = 2$. The proof for each case proceeds using a sequence of claims.

CASE 1 OF LEMMA 4.3: $|F_{u,v}| \geq 3$.

Claim 4.4 *If $|F_{u,v}| \geq 3$, then the bubbles in $F_{u,v}$ are disjoint, and there is no edge in G going from one bubble to another.*

Proof: For each bubble in $F_{u,v}$, there is a path crossing it that does not cross any other bubble in $F_{u,v}$ (otherwise we could remove that bubble and $F_{u,v}$ would not be minimal). Suppose bubbles A and B intersect. Let P_A, P_B be the paths through A and B respectively. Consider $p \in A \cap B$. Since A and B are both connected, there is a path in A from a node in $P_A \cap A$ to p and a path in B from $P_B \cap B$ to p . This creates a K_4 minor with any third path from u to v , which exists since $|F_{u,v}| \geq 3$, contradicting the series-parallelness of G ; and so A and B do not intersect. If there is an edge connecting the bubbles A and B , there is again a path in $A \cup B$ connecting P_A to P_B , which again creates a K_4 minor. ■

We contract every edge that does not cross one of the cuts defined by the bubbles in $F_{u,v}$. We get one vertex f_i for each bubble, one vertex u' for the part of the graph reachable from u without crossing the bubbles, and one vertex v' for the part reachable from v without crossing the bubbles. We prove there are no other vertices.

Claim 4.5 *The contracted supply graph is a $K_{2,m}$.*

Proof: We know that in the uncontracted graph, there is a path connecting u to v through each bubble, disjoint from the other bubbles. And so, there is an edge from u' and v' to each f_i , and these vertices induce a $K_{2,m}$ subgraph. Suppose there is another vertex x . The vertex x cannot be adjacent to u' or v' , because the edge between them would have been contracted. It cannot be connected to two different vertices f_i and f_j , because this would create a K_4 . So it is connected to a single f_i , and it is a leaf. But the set $\{f_i\}$ defines a tight cut, and since x is a leaf, the set $\{f_i, x\}$ would define a cut with a smaller surplus than $\{f_i\}$, which is not possible. So x does not exist. ■

The contracted instance has a $K_{2,m}$ supply graph. Each vertex f_i of degree 2 defines a tight cut, since it is the result of contracting a tight set. So in any fractional solution to the contracted instance, the two supply edges leaving f_i have just enough capacity to route the demands incident to f_i , and no flow can go from u' to v' through f_i . Since there is a demand from u' to v' , this means that the instance does not have a solution, and therefore, by Theorem 4.1, it contains an odd spindle as a minor.

This finishes the case $|F_{u,v}| \geq 3$.

CASE 2 OF LEMMA 4.3: $|F_{u,v}| = 2$.

Suppose $F_{u,v} = \{A, B\}$, for distinct bubbles A, B . By a sequence of claims, we prove that if we contract every edge that does not cross a bubble in $F_{u,v}$, we get an instance with a $K_{2,m}$ supply graph, satisfying the cut condition, but unroutable. Appealing to Theorem 4.1 again, we conclude that the instance contains an odd spindle as minor.

Claim 4.6 *If A and B are two bubbles covering every simple path from u to v , then A and B intersect.*

Proof: Let R be the connected component of $V \setminus (A \cup B)$ containing v . Let $X = A \cup R$ and $Y = B \cup R$. Suppose A and B are disjoint. Then $X \cap Y = R$.

Now $\sigma(X \setminus Y) = \sigma(A) = 0$, and $\sigma(Y \setminus X) = \sigma(B) = 0$. By Lemma 2.2(d), we have $\sigma(X) + \sigma(Y) = \sigma(X \setminus Y) + \sigma(Y \setminus X) + 2\sigma(X \cap Y, V \setminus (X \cup Y)) < 0$, because $\sigma(X \cap Y, V \setminus (X \cup Y))$ includes the demand (u, v) and $X \cap Y = R$ which is disconnected from the rest of the graph by A and B . However, by the cut condition, $\sigma(X) \geq 0$ and $\sigma(Y) \geq 0$, a contradiction. Therefore A and B intersect. ■

Claim 4.7 *There are two vertices, taken from $A \setminus B$ and $B \setminus A$ respectively, that form a 2-vertex-cut of G , separating it into at least three connected components, with u and v in different components.*

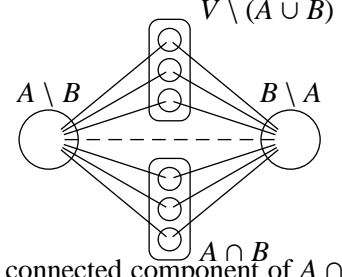
Proof: Let U be the connected component (in G) of $V \setminus (A \cup B)$ containing u , and let R be the connected component of $V \setminus (A \cup B)$ containing v . Since A is central, there is a path from u to v outside A which goes through $B \setminus A$. Symmetrically, there is a path from u to v outside B which goes through $A \setminus B$. These two paths form a cycle C going through $U, A \setminus B, R$ and $B \setminus A$ in order. By Claim 4.6, there is a vertex $x \in A \cap B$. Since x is in A , there is a path P_a in A from x to $C \cap (A \setminus B)$. Let a be the endpoint of P_a on C . Since x is in B , there is a path P_b in B from x to $C \cap (B \setminus A)$. Let b be the endpoint of P_b on C . The paths P_a and P_b only intersect in $A \cap B$. So there are three vertex-disjoint paths in G from a to b , one through U , one through R , and one through $A \cap B$. So (a, b) must be a 2-vertex-cut, for otherwise G would have a K_4 minor. ■

Claim 4.8 *The sets $A \setminus B$ and $B \setminus A$ are both central.*

Proof: By Claim 4.7, $A \setminus B$ and $B \setminus A$ contain a pair of vertices that is a vertex 2-cut separating u from v . We use Lemma 2.4 of [4], which proves that in a series-parallel graph, this implies that $A \setminus B$ and $B \setminus A$ are both central. ■

Claim 4.9 *If we contract every edge of G that is neither in $\delta_G(A)$, nor in $\delta_G(B)$, and merge parallel edges, we get a $K_{2,m}$, with possibly one extra supply edge connecting the two vertices not of degree 2.*

Proof: Since $A \setminus B$ is central, it is connected. Similarly, $B \setminus A$ is connected. The rest of the graph is composed of $A \cap B$, which has at least one connected component by Claim 4.6, and $V \setminus (A \cup B)$, which has at least two connected components containing u and v respectively. There is an edge connecting $A \setminus B$ to each connected component of $A \cap B$ (because both are in A , which is central), and there is an edge connecting $A \setminus B$ to each connected component of $V \setminus (A \cup B)$ (because neither is in B , which is central). Similarly, there is an edge connecting $B \setminus A$ to each connected component of $A \cap B$ and $V \setminus (A \cup B)$. This implies that for each connected component of $A \cap B$ and $V \setminus (A \cup B)$, there is a path connecting $A \setminus B$ to $B \setminus A$ through that component. As a consequence, there is never an edge going from a connected component of $A \cap B$ to a connected component of $V \setminus (A \cup B)$, because this would create a K_4 with $A \setminus B$ and $B \setminus A$, which are also connected through at least another connected component of $V \setminus (A \cup B)$. ■



Let us perform the contraction described in Claim 4.9. After the contraction, the endpoints of the demand edge (u, v) are still separated by the sets A and B , which are still tight. So the contracted instance is not routable, even though it satisfies the cut condition. And so, the contracted pair of graphs is not cut-sufficient. But since the contracted supply graph is a $K_{2,m}$, Theorem 4.1 implies that the contracted pair contains an odd spindle as minor. Therefore, so does the original pair (G, H) . We are now done with the case $|F_{u,v}| = 2$. This completes the proof of Lemma 4.3. ■

4.2 Identifying a bubble-covered demand

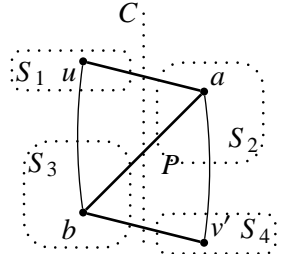
To finish the proof of Theorem 1.1, we must show that the conditions of Lemma 4.3 are satisfied, so that our instance (G, H) does have an odd spindle as a minor. The next lemma shows precisely this. The proof of this lemma uses the notions of *split pairs* and *bracketing*, defined in Section 2.2.

Lemma 4.10 *If a simple pair (G, H) has flow-cut gap greater than 1, then there is a demand (u, v) such that $\mathcal{P}[u, v]$ is covered by bubbles.*

Proof: We choose an arbitrary split pair in graph G , and orient G accordingly. By Theorem 3.1 of [4], there must be at least one non-compliant demand. We then choose a non-compliant demand (u, v) such that its pair of terminals does not strictly bracket the pair of terminals of any other non-compliant demand. This is always possible, since in the set of pairs of terminals, the bracket relation is a partial order and must have a minimal pair.

Suppose $\mathcal{P}[u, v]$ is *not* covered by bubbles. We shall demonstrate a contradiction with our choice of (u, v) . By Observation 4.2, a path in $\mathcal{P}[u, v]$ not covered by a bubble must cross some tight cut an odd number of times, more than once.

Let P_1, \dots, P_k be the paths in $\mathcal{P}[u, v]$ not covered by any bubble. For each $j \in \{1, \dots, k\}$, let C_j be the set of tight cuts that P_j crosses an odd number of times, three or more, and let m_j be the sum over all cuts $C' \in C_j$ of the number of times that P_j crosses C' . By Observation 4.2, each C_j is nonempty. We choose a path $P = P_j$ such that m_j is minimal. Let C be a cut in C_j (therefore P crosses C at least three times), and let S_1, S_2, S_3 and S_4 be the first four connected components in order of $P \setminus \delta_G(C)$, with $S_1, S_3 \subseteq C$ (see figure). Since C is central, S_1 and S_3 are connected by a path P_{13} inside of C , and S_2 and S_4 are connected by a path P_{24} outside of C . Let a be the endpoint of P_{24} in S_2 , let b be the endpoint of P_{13} in S_3 , and let v' be the endpoint of P_{24} in S_4 . Note that there are three vertex-disjoint paths from a to b , and so (a, b) is a 2-vertex-cut separating u from v' , for otherwise G would have a K_4 minor.



The proof proceeds using a sequence of claims. The following arguments use C , u and b , but apply symmetrically to $V \setminus C$, v' and a .

Claim 4.11 *Any path from u to b inside C must cross some tight cut at least twice more than P .*

Proof: If a path from u to b crosses no tight cut more than once, then we shortcut P with that path, and get a simple path P' from u to v that does not cross any bubble. Therefore $P' = P_\ell$ for some $\ell \in \{1, \dots, k\}$. Now P_ℓ crosses $\delta_G(C)$ twice less than P , and does not cross any other tight cut more times than P ; therefore $m_\ell < m_j$, contradicting the minimality of m_j . ■

Recall that (a, b) is a 2-vertex-cut separating u from v' . Let S_u and S_v be the connected components of $V \setminus \{a, b\}$ containing u and v' , and let $S_u^* = S_u \cup \{a, b\}$ and $S_v^* = S_v \cup \{a, b\}$. For subsets $S, C \subseteq V$ and vertices $u, b \in C$, we say that S separates u from b inside C if u and b are in two different connected components of $C \setminus S$.

Claim 4.12 *There is a 2-vertex-cut (x, y) in S_u^* , with not both x and y in $\{a, b\}$, with two vertex-disjoint paths Q_1, Q_2 from x to y , with $Q_1 \setminus \{x, y\}$ and $Q_2 \setminus \{x, y\}$ not containing a or b , and a demand i from $Q_1 \setminus \{x, y\}$ to $Q_2 \setminus \{x, y\}$.*

Proof: Since C is central, u and b are connected inside C , and by Claim 4.11, any path from u to b crosses some tight cut at least twice more than P . Either there is a single tight cut crossed by all such paths, or there is not. We prove these separate cases in Claim 4.13 and Claim 4.14 respectively. ■

Claim 4.13 *If all paths from u to b inside C cross twice the same tight cut, then there is a vertex $x \in Q$ separating u from b inside C , and a vertex $y \in P \setminus C$, such that x and y are connected by two vertex-disjoint paths Q_1 and Q_2 that do not contain b , with a demand edge going from some vertex in $Q_1 \setminus \{x, y\}$ to some vertex in $Q_2 \setminus \{x, y\}$. Either y is a , or y is in the connected component of $V \setminus \{a, b\}$ that contains u .*

Proof: Let S be the central set defining the tight cut crossed twice by all paths, with S containing neither u nor b . Since S is not crossed by P on the way from u to b , S does not contain a ; and since the pair (a, b) is a 2-vertex-cut separating u from v' , S does not contain v' .

Let U be the connected component of $C \setminus S$ containing u . Let $B = C \setminus (S \cup U)$. By Lemma 2.2⟨e⟩, $\sigma(U) - 2\sigma(S, U) = \sigma(S \cup U) - \sigma(S) \geq 0$. Then by Lemma 2.2⟨a⟩,

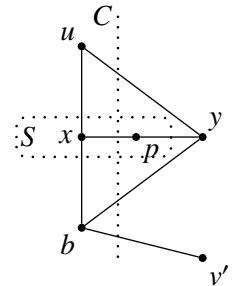
$$\sigma(U) \geq 2\sigma(S, U) = 2\sigma(S \cap C, U) + 2\sigma(S \setminus C, U). \quad (4)$$

Since P is not covered by a bubble, $\sigma(C \setminus U) > 0$. By Lemma 2.2⟨e⟩, since $U \subseteq C$, $\sigma(U) - 2\sigma(C \setminus U, U) = \sigma(C) - \sigma(C \setminus U) < 0$. Then by Lemma 2.2⟨a⟩,

$$\sigma(U) < 2\sigma(C \setminus U, U) = 2\sigma(S \cap C, U) + 2\sigma(B, U) \quad (5)$$

Subtracting (4) from (5), we get that $\sigma(S \setminus C, U) < \sigma(B, U)$. Since there is no supply edge from U to B , $\sigma(B, U) \leq 0$, which proves that there is a demand from some vertex $q \in U$ to some vertex $p \in S \setminus C$. Since the subpath of P from u to b has vertices outside C , there is a path Q' connecting p to P outside of C . Let y be the endpoint of Q' in P . Since Q intersects S , there must be a path Q'' connecting p to Q inside S . Let x be the endpoint of Q'' in Q . The paths P and Q form a cycle containing the vertices u, y, b and x . The paths Q' and Q'' form a path from x to y disjoint from that cycle, and so there are three vertex-disjoint paths from x to y , and (x, y) is a 2-vertex-cut separating u, p and b , otherwise there would be a K_4 . So x separates u from b in C .

Recall that there is a demand from p to some vertex $q \in U$. By Lemma 2.4, there is a simple path Q_1 from x to y containing p , and a simple path Q_2 from x to y containing q . The paths Q_1 and Q_2 must be



vertex-disjoint; otherwise there would be a path from p to q disjoint from $\{x, y\}$, and since U is a connected component of $C \setminus S$, a path inside U from q to u disjoint from $\{x, y\}$, contradicting the fact that (x, y) separate p from u . Finally, (a, b) is a 2-vertex-cut, so since there is a path from u to y through x disjoint from $\{a, b\}$, y cannot be in a different connected component of $V \setminus \{a, b\}$ than u . ■

Claim 4.14 *If there is no single tight cut crossed twice by all paths connecting u and b inside C , then there is a demand edge going from one of those paths to another.*

Proof: For every path connecting u to b inside C , we choose a tight cut crossed twice, and we contract all edges of the path that do not cross that tight cut. Each of the paths now has two edges. Let S denote the set of vertices in the middle of these paths. There are no supply edges from a vertex in S to any vertex except u or b because that would create a K_4 minor. Since every path from u to b crosses some central tight cut twice, every vertex in S defines a bubble separating u from b inside C . The supply graph induced by u , b and vertices in S is a $K_{2,m}$. By assumption, there is no single tight cut crossed twice by all paths, so $\sigma(S) > 0$, even though every vertex in S defines a tight cut. And so, by Lemma 2.2(e), there must exist demands between vertices of S . ■

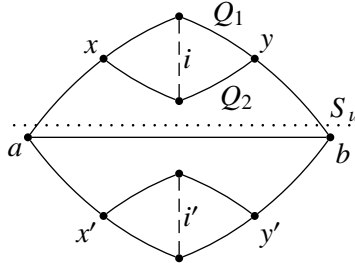
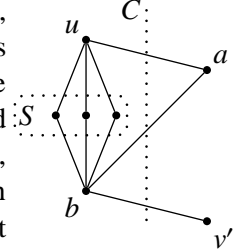


Figure 2: Subgraph showing the relations of the 2-vertex-cut (a, b) , (x, y) and (x', y') . One of x or y may be a or b , but not both. One of x' or y' may be a or b , but not both. The demands i and i' are dashed.

Note that Claim 4.12 also applies to S_v^* , and so there is in S_v^* a 2-cut (x', y') , with two vertex-disjoint paths from x' to y' , and a demand i' connecting these two paths (see Figure 2).

Recall that (s, t) is a split pair. Since (a, b) is a 2-vertex-cut connected by three disjoint paths, s and t cannot be in different connected components of $V \setminus \{a, b\}$, because otherwise an (s, t) edge would create a K_4 . So at least one of S_u or S_v contains neither s nor t .

Claim 4.15 *Suppose S_u contains neither s nor t . Then i is non-compliant; the pair (x, y) which separates its endpoints is its pair of terminals, and this pair of terminals is strictly bracketed by the pair (w, z) of terminals of (u, v) .*

Proof: By Lemma B.4, for any $v' \in S_u$, there is a simple path from s to t containing v' , so there is a simple path from s to t that goes through S_u , and so contains a and b . Without loss of generality, assume that the path meets a before b on the way from s to t . Then since the orientation is acyclic, there is no simple path from s to t that meets b before a . Since any edge in G is oriented in the direction it appears on any simple path from s to t , then any edge in S_u^* is oriented in the direction it appears on any simple path from a to b . So a is the unique source in S_u^* , and b the unique sink. Any simple path from a to b through an endpoint of i contains x and y , and does not contain the other endpoint of i . So i is a non-compliant demand, and (x, y) is its pair of terminals, which is bracketed by (a, b) . Note that (x, y) is not the same as (a, b) .

We prove that (a, b) is bracketed by the pair (w, z) of terminals of (u, v) , which means that (x, y) is bracketed by (w, z) . By Lemma 2.7, any cycle C containing u and v also contains the terminals w and z of

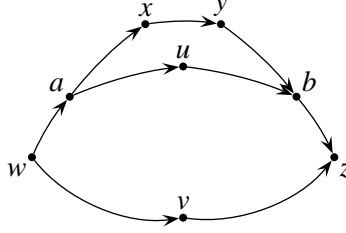


Figure 3: Relative positions of w, z, a, b, x, y, u and v . It is possible that $w = a$, or $z = b$. The vertex u may be on the path from a to b containing x and y , but does not need to be.

the demand (u, v) , and is composed of two oriented paths from one terminal to the other, say from w to z , and w is the unique source of C and z its unique sink. The cycle C must contain a and b since (a, b) is a 2-vertex-cut separating u from v . Since any simple path from a to b in S_u^* is oriented from a to b , the part of C in S_u^* is oriented from a to b . So neither w nor z is in S_u , because then they would not be source or sink of C . So C contains a path Q from w to z through u , and Q contains a and b ; so (a, b) is bracketed by (w, z) . So (x, y) is bracketed by (w, z) . ■

Since at least one of S_u and S_v contains neither s nor t , at least one of (x, y) and (x', y') is bracketed by (w, z) , contradicting our choice of (u, v) . This completes the proof of Lemma 4.10. ■

5 Integrally Routable Series-Parallel Instances

In this section, we prove Theorem 1.2, which we restate here:

Theorem 5.1 *Let (G, H, c, D) form an instance of the multicommodity flow problem, such that G is series-parallel, (G, H) is cut-sufficient, and (G, H, c, D) is Eulerian. Then the instance has an integral solution if and only if it satisfies the cut condition, and that integral solution can be computed in polynomial-time.*

Since an instance that does not satisfy the cut condition cannot have a solution, integral or otherwise, we only need to prove the other direction.

For any demand $d = (u, v)$ and vertex w in a multiflow instance, *pushing a unit of d to w* consists of removing one unit of demand d , and creating two demand edges of unit demand from u to w and w to v . This can be seen as taking the decision of routing at least one unit of the demand d through w .

For any demand d whose endpoints are connected by a path P , *routing a unit of d along P* consists of removing one unit of capacity along each edge of P , and removing one unit of demand from d . Supply edges whose capacity falls to zero are removed from G , and demand edges whose demand falls to zero are removed from H . For each $S \subseteq V$, define $n_S = |\delta_G(S) \cap P|$. The operation reduces the surplus $\sigma(S)$ by $2\lfloor n_S/2 \rfloor$: it reduces the total of capacities crossing $\delta_G(S)$ by n_S ; and if n_S is odd, then $d \in \delta_H(S)$ and it reduces the total demand crossing $\delta_H(S)$ by 1. Thus, the surplus of any cut is reduced by an even number.

Suppose we are given a series-parallel instance that is cut-sufficient, Eulerian, and satisfies the cut condition, with a demand $d = (u, v)$. We prove that (A) there is a sequence of push operations to move a unit of demand d to a path Q of unit demands from u to v without breaking the cut condition; and (B) the unit demands in Q can all be routed without breaking the cut condition. Thus, the demands in Q fall to zero, and are removed. The two operations are equivalent to routing one unit of d ; thus, we get a smaller instance which has the same properties. We can therefore recursively build a solution to the whole problem.

We embed G in the plane such that the endpoints u and v are on the outside face. (Lemma 2.8) Any path P from u to v thus partitions $G \setminus P$ into two sides, one to the left and one to the right of P . Two paths P and P' cross if P' contains vertices on both sides of P . We decompose the flow of the fractional solution routing the

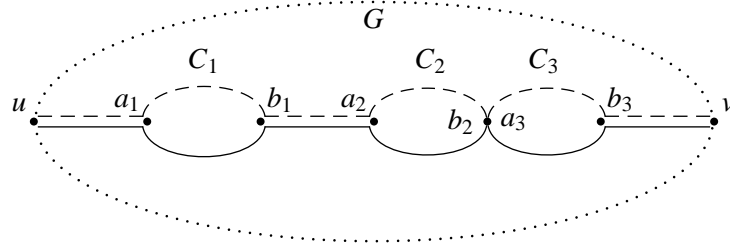


Figure 4: Illustration of the planar embedding with u and v on the outside face. The dotted cycle represents the outside face of G , the paths P_1 is in solid, and the path P_k in dashed.

demand d into paths in the series-parallel supply graph such that no two paths cross. This gives an ordering of the path P_1, \dots, P_k such that if P_1 and P_k have a common vertex, then all paths P_j , $j = 1, \dots, k$ go through that vertex. We examine the subgraph $P_1 \cup P_k$. Since u and v are on the outside face of G , the graph $P_1 \cup P_k$ is composed of a family of cycles (whenever P_1 and P_k are disjoint) connected by paths (whenever P_1 and P_k coincide). Let C_1, \dots, C_j be the cycles in $P_1 \cup P_k$, and for any cycle C_i , let a_i and b_i be the two vertices of C_i contained in both P_1 and P_k . See Figure 4.

Lemma 5.2 *In any instance of the multiflow problem satisfying the cut condition, if there is a fractional solution such that all paths P_1, \dots, P_k routing demand $d = (u, v)$ go through the same vertex w , then it is possible to push a unit of the demand d to the vertex w without breaking the cut condition.*

Proof: Let $C_{uv,w}$ be the set of cuts separating u and v from w . If we push a unit of d to w , only the surpluses of cuts in $C_{uv,w}$ are modified, and each surplus is reduced by two units. It is thus sufficient to prove that all cuts in $C_{uv,w}$ have a surplus of at least two.

We execute the following operations on the multiflow problem and its fractional solution. We reduce the demand of d by one unit. Let f_1, \dots, f_k be the flows of the fractional solution routed on paths P_1, \dots, P_k . We chose quantities $0 \leq g_i \leq f_i$, $i = 1, \dots, k$, such that $\sum_i g_i = 1$. We remove successively from each edge in P_i a quantity g_i of capacity, and subtract g_i from f_i , with $i = 1, \dots, k$.

Since each path P_i crosses every cut in $C_{uv,w}$ at least twice, these operations reduce the surplus of every cut in $C_{uv,w}$ by at least two. The remainder flow of f_1, \dots, f_k on paths P_1, \dots, P_k gives a fractional solution routing the reduced demand, and so the instance still satisfies the cut condition. So for each $S \in C_{uv,w}$, $\sigma(S) \geq 0$ after $\sigma(S)$ has been reduced by at least two, so $\sigma(S)$ was at least two in the original instance. ■

We push the demand d to every vertex in $P_1 \cap P_k$. By Lemma 5.2, we can do this without breaking the cut condition, since all paths routing d in the fractional solution go through these vertices. This creates a path Q of unit demands from u to v , such that the vertices of Q are the vertices in both P_1 and P_k . This completes part (A).

We next argue that we can route the demands in Q . We need to identify paths in G to do this. The path Q has a unit demand parallel to every edge in the paths connecting the cycles C_1, \dots, C_j , and a unit demand from a_i to b_i for every cycle C_i , $i = 1, \dots, j$. We will route the demands in Q along the paths connecting the cycles, and then along one side of each cycle. The side we pick is guided by the next two lemmas.

For any cycle C_i not containing v , we say a vertex $w \in C_i$ is *linked to v* if there is in G a path from w to v containing only the vertex w in C_i .

Lemma 5.3 *Any cycle C_i not containing v contains at most one vertex apart from b_i that is linked to v .*

Proof: Contract the connected component of $G \setminus C_i$ containing v . The resulting vertex is connected by an edge to any vertex of C_i that is linked to v . If there are three, this forms a K_4 . ■

We define the path P from u to v by choosing for each cycle C_i the side of C_i from a_i to b_i that does not contain a vertex linked to v . This is always possible by Lemma 5.3.

If $\delta_G(S)$ is a central cut, by Lemma 2.3 it crosses a cycle C_i either twice, or not at all. For any u -to- v path P' in $P_1 \cup P_k$ obtained by choosing for each cycle C_i either $C_i \cap P_1$ or $C_i \cap P_k$, the cut $\delta_G(S)$ crosses $P' \cap C_i$ zero, once, or twice for every $i = 1, \dots, j$. Our choice of P given above is special:

Lemma 5.4 *For any central cut $\delta_G(S)$, there is at most one cycle C_i such that $\delta_G(S)$ crosses $P \cap C_i$ twice.*

Proof: Suppose that there is a set S defining a cut that crosses $P \cap C_i$ twice and $P \cap C_l$ twice, for $i < l$. Then S either contains both a_i and b_i , or neither of them. Suppose without loss of generality that it contains neither. Then S contains some vertices in $P \cap C_i \setminus \{a_i, b_i\}$. Since $\delta_G(S)$ also intersects C_l , the set S also contains some vertex in C_l . As S is central, there must be a path from $(P \cap C_i) \setminus \{a_i, b_i\}$ to C_l , which means that some vertex of $(P \cap C_i) \setminus \{a_i, b_i\}$ is linked to v . This contradicts our choice of P . ■

Lemma 5.5 *We can route the unit demands in Q along the path P without breaking the cut condition.*

Proof: The path P goes through both extremities of every demand we created by pushing d . Routing any demand parallel to a supply edge consists of removing one unit of capacity from the supply edge and removing the unit demand. The surplus of any cut crossing such a demand is not affected by this. Routing a demand across a cycle C_i , from a_i to b_i , consists of removing one unit of capacity of each supply edge in $P \cap C_i$, and removing the unit demand. If a central cut $\delta_G(S)$ separates a_i from b_i , it crosses $P \cap C_i$ exactly once, and so its surplus $\sigma(S)$ is not affected by this. If a central cut $\delta_G(S)$ does not separate a_i from b_i , then its surplus is reduced by two or unchanged, depending on whether it crosses $P \cap C_i$ twice or not at all. For any central cut $\delta_G(S)$, there is at most one cycle C_i such that $\delta_G(S)$ crosses $P \cap C_i$ twice, by Lemma 5.4. So the surplus of any cut is reduced at most by two. As there is a positive flow routing demand d along path P in the fractional solution, no cut that crosses P more than once is tight: because in any solution to the multiflow problem, the supply edges crossing a tight cut have their capacity completely used to route the demands that also cross it. As the instance is Eulerian, any cut that is not tight has a surplus of at least two. And so routing one unit along P does not break the cut condition. ■

The flow routing all the demands created by pushing d is also a way of routing one unit of $d = (u, v)$ in the original problem; so we have found a path P from u to v such that routing one unit of d along this path does not break the cut condition. After doing this, the reduced instance still does not have any odd spindle as a minor, since no demand edges were introduced; is still Eulerian, and still satisfies the cut condition. By induction, we can find an integral routing for the instance.

5.1 Polynomial-Time Algorithm

The method described in the proof of Theorem 5.1 routes one unit of flow at a time. We first show that each unit can be routed in polynomial-time. This gives us a pseudo-polynomial-time algorithm for an instance (G, H, c, D) ; the algorithm is polynomial in the size of $G = (V, E)$, $H = (V, F)$ and the bit-size of c , but only polynomial in D , the demands assigned to edges of H , instead of in the *bit-size* of D . We then give a fully-polynomial-time algorithm, that reduces the instance to another one in which D is polynomial in the size of G and H , and then uses the pseudo-polynomial-time algorithm.

First, it is possible to find a fractional solution to the problem in polynomial-time by linear programming. The problem can indeed be solved by a polynomial-sized linear program, by having one variable f_e^i indicating the amount of commodity i flowing through edge e , for every $i \in F$ and $e \in E$ (e.g. Section 70.6 of [17]). This linear program can be then solved efficiently in polynomial time using interior point methods.

The second step is to embed the planar graph G' into the plane. This can be done in time linear in the number of vertices [15].

We then decompose the flow of the fractional solution routing a demand into paths P_1, \dots, P_k . Let $m = |E|$. The flow decomposition has $k \leq m$ paths, and can be found in $O(m^2)$ time, given the fractional flow.

Finally, we find for each of the $O(m)$ cycles in $P_1 \cup P_k$ which side has a vertex linked to v . This can be done by an exploration algorithm in $O(m)$ time, which makes $O(m^2)$ time in total. The operation of routing a unit through the path P is done in $O(m)$ time.

So routing one unit of demand can be done in polynomial-time, with a theoretical complexity dominated by the resolution of the linear program finding a fractional solution.

We now present a polynomial-time algorithm. We start by finding a fractional solution to the problem, solving the polynomial-sized linear program. For each demand $i \in F$, we do a path decomposition of the flow routing i . This yields $k \leq m$ paths P_1, \dots, P_k per demand i . For each path P routing a quantity f_P^i of flow between endpoints of i , we send $\lfloor f_P^i \rfloor$ units of flow on P . After this, each path P_j routes an amount of flow smaller than 1, and since there are no more than m paths routing each demand, we are left with at most $m|F|$ units of demand to route. We use then the pseudo-polynomial algorithm presented above. The theoretical complexity of the algorithm is dominated by that of this last step, which solves at most $m|F|$ linear programs finding a fractional solution.

6 Discussion

In this paper, we give a complete characterization for cut-sufficient multiflow problems in series-parallel instance. A pair (G, H) is *minimally cut-insufficient* if it is not cut-sufficient, but deleting any edge or demand or contracting any edge makes it cut-sufficient. Since any pair that is not cut-sufficient contains a pair that is minimally cut-insufficient as a minor, then our results show that odd spindles are the only minimally cut-insufficient pairs with G series-parallel.

A natural extension of this result is to planar pairs, i.e., pairs where the supply graph is planar. There are planar pairs that are not cut-sufficient, yet do not have an odd spindle as a minor. A *bad- K_4 -pair* is the example in Figure 5, attributed by [17] to Papernov, which is of particular interest. Apart from odd spindles, it is the only minimally cut-insufficient pair we know of.

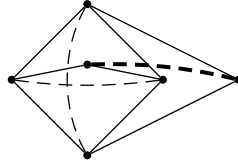


Figure 5: Planar pair without odd spindle as a minor, and not cut-sufficient. Supply edges are solid, and demands are dashed. If the thick dashed edge has demand 2, and all other capacities and demands are 1, the instance is Eulerian and satisfies the cut condition, but is not routable.

Conjecture 6.1 *Odd spindles and the bad- K_4 -pair are the only minimally cut-insufficient pairs (G, H) , with G planar.*

This would imply that a planar pair is cut-sufficient if and only if it does not contain an odd spindle or the bad- K_4 -pair in Figure 5 as a minor.

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